Exams for Matrix Computations

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1 Problem 1

For each $\mathbf{A} \in \mathbb{R}^{m\times n}$, prove that its Moore-Penrose inverse exists and is unique.

2 Problem 2

Let $\mathbf{A}$ and $\mathbf{B} \in \mathbb{R}^{n\times n}$ be symmetric. Suppose that $\mathbf{B}$ is positive definite. Prove that if λ_1 and λ_n are the largest and smallest eigenvalues of $\mathbf{B}^{-1}\mathbf{A}$, then

$$\lambda_n \leq \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{B}\mathbf{x}} \leq \lambda_1 \quad \text{for any } \mathbf{x} \neq 0.$$

3 Problem 3

Let $\mathbf{A} = [a_{ij}]$ be a $p\times q$ matrix and $\mathbf{B} = [b_{ij}]$ be an $r\times s$ matrix. The Kronecker product of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A}\otimes\mathbf{B}$, is the $pr \times qs$ matrix

$$\mathbf{A}\otimes\mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1q}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2q}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1}\mathbf{B} & a_{p2}\mathbf{B} & \cdots & a_{pq}\mathbf{B} \end{bmatrix}.$$

Prove

(a) If $\mathbf{A}$ is $m\times n$, $\mathbf{B}$ is $p\times q$, $\mathbf{C}$ is $n\times r$, and $\mathbf{D}$ is $q\times s$, then

$$(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}) = (\mathbf{AC})\otimes(\mathbf{BD}).$$

(b) $(\mathbf{A}\otimes\mathbf{B})^T = \mathbf{A}^T\otimes\mathbf{B}^T$.

(c) If $\mathbf{A}$ and $\mathbf{B}$ are both $m \times m$, then

$$\mathrm{tr}(\mathbf{A}\otimes\mathbf{B}) = \mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B}).$$

(d) If $\mathbf{A}$ and $\mathbf{B}$ are nonsingular, then

$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}.$$

(e) If $\mathbf{A}$ is $m \times m$ and $\mathbf{B}$ is $n \times n$, then

$$\det(\mathbf{A} \otimes \mathbf{B}) = (\det(\mathbf{A}))^n (\det(\mathbf{B}))^m.$$

4 Problem 4

Let $\mathbf{A}$ be real and antisymmetric (i.e., $\mathbf{A}^T = -\mathbf{A}$). Show that $\mathbf{I} - \mathbf{A}$ and $\mathbf{I} + \mathbf{A}$ are both nonsingular and that $(\mathbf{I} - \mathbf{A})^{-1}(\mathbf{I} + \mathbf{A})$ is orthogonal.

5 Problem 5

Show that $||\mathbf{x}||_1 ||\mathbf{x}||_\infty \leq \frac{1+\sqrt{n}}{2} ||\mathbf{x}||_2^2$ for any $\mathbf{x} \in \mathbb{R}^n$.

6 Problem 6

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ have eigenvalues $\lambda_1, \lambda_2 \ldots, \lambda_n$ and singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$. Prove that

$$||\mathbf{A}||_F^2 = \sum_{i=1}^{n} \sigma_i^2 \geq \sum_{i=1}^{n} |\lambda_i|^2,$$

with equality if and only if $\mathbf{A}$ is normal.

7 Problem 7

For $\mathbf{A} \in \mathbb{R}^{m \times n}$, let $q = \min\{m, n\}$, and let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_q$ be the singular values of $\mathbf{A}$. Prove

(a) $\sum_{j=1}^{q} \sigma_j$ is a matrix norm on $\mathbb{R}^{m \times n}$;

(b) $\sum_{j=1}^{q} \sigma_j$ is not a matrix operator norm on $\mathbb{R}^{m \times n}$.